Some Generic Properties of Level Spacing Distributions of 2D Real Random Matrices

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We study the level spacing distribution $P(S)$ of 2D real random matrices both symmetric as well as general, non-symmetric. In the general case we restrict ourselves to Gaussian distributed matrix elements, but different widths of the various matrix elements are admitted. The following results are obtained: An explicit exact formula for $P(S)$ is derived and its behaviour close to $S = 0$ is studied analytically, showing that there is linear level repulsion, unless there are additional constraints for the probability distribution of the matrix elements. The constraint of having only positive or only negative but otherwise arbitrary non-diagonal elements leads to quadratic level repulsion with logarithmic corrections. These findings detail and extend our previous results already published in a preceding paper. For *symmetric* real 2D matrices also other, non-Gaussian statistical distributions are considered. In this case we show for arbitrary statistical distribution of the diagonal and non-diagonal elements that the level repulsion exponent ρ is always $\rho = 1$, provided the distribution function of the matrix elements is regular at zero value. If the distribution function of the matrix elements is a singular (but still integrable) power law near zero value of S, the level spacing distribution $P(S)$ is a fractional exponent power law at small S. The tail of $P(S)$ depends on further details of the matrix element statistics. We explicitly work out four cases: the uniform (box) distribution, the Cauchy-Lorentz distribution, the exponential distribution and, as an example for a singular distribution, the power law distribution for $P(S)$ near zero value times an exponential tail.

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1. Introduction

Random matrix theory [1 – 5] has important applications quite generally in the statistical description of complex systems as e. g. for complex nuclei, for which it has been originally developed, or for chaotic systems with just a few degrees of freedom as treated in quantum chaos. Usually one restricts oneself to Gaussian ensembles of random matrices, meaning that the matrix elements have a Gaussian distribution where the diagonal matrix elements have all the same dispersion, the off-diagonal elements also have the same dispersion, but the former is by a factor 2 larger than the latter one, see the remarks in Section 2. This ensemble is the only one which is invariant under symmetry transformations of the underlying matrix **A**. If **A** is real and symmetric, the group of relevant transformations consists of the orthogonal transformations and we speak of the Gaussian Orthogonal Ensemble (GOE), while for Hermitian **A** the relevant group of transformations is that of the unitary transformations and we speak of the Gaussian Unitary Ensemble (GUE) of random matrices. The property of the matrix element distribution to be Gaussian is a direct consequence of just two assumptions, namely statistical independence of the distributions of the matrix elements and invariance against the group of appropriate transformations. We usually have in mind infinite dimensional matrices, although for many purposes finite dimensionality is useful and sufficient.

Several generalizations are possible. One is the generalization towards general non-normal matrices, either fully complex [6] (see also [1]) but still Gaussian (invariant), or real positive but no longer invariant against the above group of transformations while still Gaussian (with different variances for different matrix elements) [7]. In the latter paper [7] we have also treated 2D real symmetric matrices whose matrix elements are Gaussian distributed but with different variances of the diagonal and the non-diagonal matrix elements. Thus they no longer enjoy GOE invariance, but still have Gaussian distributed matrix elements.

In the present paper we study the level spacing distribution $P(S)$ of 2D real random matrices by considering general distributions of the matrix elements, going beyond Gaussian. Therefore these ensembles of random matrices are no longer invariant under the mentioned transformation groups. The statistics $P(S)$ of the level spacings S changes under transformations with the elements of those groups, so it depends on the basis chosen for their representation. Nevertheless, they are important in specific physical situations and also involve interesting mathematics.

We shall first treat rigorously the case of general 2D non-normal matrices[1] with Gaussian distributed matrix elements even admitting different variances of the various matrix elements. The level repulsion exponent ρ in this case will turn out to be still $\rho = 1$, unless there are constraints on the matrix elements like e. g. having only positive or only negative non-diagonal elements. More general matrix element distributions in this case are left for future research, as they introduce additional serious mathematical problems, cf. Section 2.

Second we consider *symmetric* matrices with other, general distributions of the real matrix elements. For such symmetric, real matrices we shall show that the level repulsion exponent is always $\rho = 1$, provided the distributions of all matrix elements are regular at zero value. Next we study in detail and without approximations the following specific cases of such regular distributions, namely: the uniform (box) distribution, the Cauchy-Lorentz distribution, the exponential distribution, and also some addenda to the Gaussian case, already dealt with in [7].

If, in contrast, the distribution of matrix elements is singular at zero value, $P(S)$ shows different behaviour at $S = 0$. For example, if the singularity of the statistics of the matrix elements is an integrable power law at zero value of S, the level spacing statistics $P(S)$ exhibits a fractional exponent power law level repulsion as was discovered and treated in [13 – 16]. This probably is characteristic for sparsed matrices, which in turn are important random matrix models for nearly integrable systems of the KAM type.

[1] A matrix $\mathbf{A}$ is non-normal [8] if it does not commute with its adjoint, i. e., $[\mathbf{A},\mathbf{A}^+\underline{]} \neq 0$. Non-normal matrices have important physical applications, especially in dissipative systems [7, 9 – 12].

2. General Real 2D Matrices

2.1. Level Spacing Statistics

Consider 2×2 matrices $\mathbf{A} = (A_{ij})$, where $i, j = 1$ or 2. This matrix has two diagonal elements, which can always be chosen as a and $-a$. For general A_{ij} introduce $A_s = \frac{1}{2}(A_{11} + A_{22})$ and subtract the diagonal matrix $A_s\underline{\mathbf{1}}$. Then $A_{11} - A_s \equiv a = -(A_{22} - A_s)$, i. e., one obtains the formula (1) without loss of generality. Quite generally, for a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ the level spacing $S = |\lambda_1 - \lambda_2| = |\sqrt{(a-d)^2 + 4bc}|$ only depends on the difference $a - d$, so that we can arbitrarily shift a and d by a constant, in particular by A_s.[2]

Let a as well as the non-diagonal elements b_1 and b_2 all be real and write

$$\mathbf{A} = (A_{ij}) = \begin{pmatrix} a & b_1 \\ b_2 & -a \end{pmatrix}. \tag{1}$$

If $b_1 = b_2$, the matrix $\mathbf{A}$ is symmetric.

The eigenvalues of $\mathbf{A}$ follow from

$$|\mathbf{A} - \lambda\underline{\mathbf{1}}| = \begin{vmatrix} a-\lambda & b_1 \\ b_2 & -a-\lambda \end{vmatrix} = \lambda^2 - a^2 - b_1 b_2 = 0, \tag{2}$$

i. e.,

$$\lambda_{1,2} = \pm\sqrt{a^2 + b_1 b_2}. \tag{3}$$

The eigenvalues are real for arbitrary symmetric ($b_1 = b_2$) matrices. In the more general case of $b_1 \neq b_2$ the eigenvalues are still real, if the product $b_1 b_2$ is larger than $-a^2$, otherwise they are purely imaginary, but never general complex.

If the matrix $\mathbf{A}$ is **not** symmetric, it is no longer normal. Namely, in the general case one finds for the commutator

$$[\mathbf{A},\mathbf{A}^+\underline{]} = \begin{pmatrix} b_1^2 - b_2^2 & 2a(b_2 - b_1) \\ 2a(b_2 - b_1) & b_2^2 - b_1^2 \end{pmatrix}. \tag{4}$$

Apparently the commutator $[\mathbf{A},\mathbf{A}^+\underline{]} = 0$ is zero iff $b_2 = b_1$, i. e., in the symmetric case. In general, 2×2 matrices $\mathbf{A}$ are non-normal, $[\mathbf{A},\mathbf{A}^+\underline{]} \neq 0$.

[2] Let us make clear that in the general symmetric GOE matrix $\begin{pmatrix} a & b \\ b & d \end{pmatrix}$ the variances of the diagonal elements a and d are equal, but by a factor 2 larger than the variance of the off-diagonal element b. However, setting $d = -a$ implies, that the GOE case occurs when the variance of a is equal to the variance of b. See also Subsection 4.1.

The distribution $P(S)$ of the level spacing S is given by

$$P(S) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \mathrm{d}a\,\mathrm{d}b_1\,\mathrm{d}b_2 \cdot \delta\left(S - 2\left|\sqrt{a^2+b_1b_2}\right|\right) g_a(a) g_{b_1}(b_1) g_{b_2}(b_2). \tag{5}$$

Here $\delta(.)$ is the Dirac delta function and $g_a(a)$, $g_{b_1}(b_1)$, $g_{b_2}(b_2)$ are the normalized probability density functions for the matrix elements a, b_1, b_2, respectively. $P(S)$ is the central object of our study in this paper. We are going to study the dependence of the main properties of $P(S)$, especially the small-S behaviour (the level repulsion) as well as the asymptotic behaviour at large S (the tail of $P(S)$), upon the main features of the matrix element distribution functions $g_a(a)$, $g_{b_1}(b_1)$, $g_{b_2}(b_2)$. In the special case of an ensemble of random symmetric matrices $\mathbf{A} = \mathbf{A}^+$ we must have $b_1 = b_2$. It is not enough that the two statistics g_{b_1} and g_{b_2} are equal! This is achieved by inserting in the integrand of (5) the constraint $g_{b_2}(b_2) = \delta(b_1 - b_2)$. Integrating then over b_2 results in the level distribution formula

$$P(S) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \mathrm{d}a\,\mathrm{d}b_1\,\delta\left(S - 2\sqrt{a^2+b_1^2}\right) \cdot g_a(a) g_{b_1}(b_1). \tag{6}$$

We shall work out exact formulae for the following cases: (i) General, non-normal matrices with Gaussian distributed elements a, b_1, b_2 in Sections 2 and 3, and (ii) symmetric (normal) matrices but considering non-Gaussian distributions of the matrix elements in Section 4: uniform (constant or box) distribution, Cauchy-Lorentz distribution, exponential distribution, and singular distribution (integrable power law at $S = 0$ multiplied by an exponential tail). Here we also detail more on the Gaussian case in order to extend our results of [7]. In Section 5 we comment on the level distribution of the prototype non-normal matrix, the triangular matrix. The final Section 6 is devoted to a discussion and conclusions.

2.2. *Polar Coordinate Representation of the Level Spacing Distribution*

We notice that the radicands in the arguments of the delta functions in both equations (5) and (6) are homogeneous in the moduli of a, b_1, and b_2. Thus it is natural to introduce spherical or plane polar coordinates. In the general case, described by (5), we define

$$b_1 = r\cos\theta\cos\varphi,\quad b_2 = r\cos\theta\sin\varphi,\quad a = r\sin\theta, \tag{7}$$

where $r \in [0,\infty)$, $-\pi/2 \le \theta \le \pi/2$, and $0 \le \varphi \le 2\pi$. Then we get for the level distance

$$S = 2\left|\sqrt{a^2+b_1b_2}\right| = 2rQ \quad \text{with}\quad Q(\theta,\varphi) = \left|\sqrt{\sin^2\theta + \frac{1}{2}\cos^2\theta\sin 2\varphi}\right|. \tag{8}$$

The Jacobian of the coordinate transformation is $r^2\cos\theta$ and therefore $\mathrm{d}a\,\mathrm{d}b_1\,\mathrm{d}b_2 = r^2\mathrm{d}r\cos\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi$. The r integration can be carried out in favour of S resulting in

$$P(S) = \frac{S^2}{8}\int_{-\pi/2}^{\pi/2}\int_0^{2\pi} \frac{\cos\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi}{Q^3} g_a\left(\frac{S}{2Q}\sin\theta\right) \cdot g_{b_1}\left(\frac{S}{2Q}\cos\theta\cos\varphi\right) g_{b_2}\left(\frac{S}{2Q}\cos\theta\sin\varphi\right). \tag{9}$$

Now, if the value of the double integral is regular at $S = 0$, the level repulsion would be quadratic, $P(S) \propto S^2$. But Q has zeros at $\theta = 0$ together with $\varphi = 0$, $\pi/2$, π, $3\pi/2$, 2π, which are scanned in the integration. That leads to singularities of the integrand, explicit ones ($1/Q^3$) as well as implicit ones (in the g arguments). In particular the Q^{-1} in the g's enforces a scan of the full matrix element distribution functions including their tails, for any non-zero value of S. One cannot simply expand the g's in terms of S and so find the small S behaviour. Thus the picture of the S dependence is far from simple. The small S behaviour depends on all details of the matrix element distribution functions g including their tails. Therefore in this general case of independent a, b_1, b_2 we shall choose another approach to attack this problem and analyse it at least in the case of Gaussian g's in Section 3.

In the case of real symmetric 2D matrices the transformation to plane polar coordinates is simpler and thus much more useful. Here the r integration does not produce singularities, since the analog of Q here is just 1. Introduce plane polar coordinates into (6),

$$a = r\cos\varphi,\quad b_1 = r\sin\varphi, \tag{10}$$

where $r \in [0,\infty)$ and $0 \le \varphi \le 2\pi$. The Jacobian is r, i. e., $\mathrm{d}a\,\mathrm{d}b_1 = r\mathrm{d}r\,\mathrm{d}\varphi$. The level distance reduces to $2\sqrt{a^2+b_1^2} = 2\,r$, which is independent of φ in contrast to the θ-, φ-dependent factor Q in the general case of (8). We now can do the r integration immediately and get

$$P(S) = \frac{S}{4}\int_0^{2\pi} \mathrm{d}\varphi\, g_a\left(\frac{S}{2}\cos\varphi\right) g_{b_1}\left(\frac{S}{2}\sin\varphi\right). \quad (11)$$

Here, for $S \to 0$ one *can* use the power law expansions of the matrix element distribution functions g_{a,b_1} in terms of S, independent of the nature of the g's, Gaussian or non-Gaussian. In particular, if $g_a(x)$ and $g_{b_1}(x)$ are both regular at $x = 0$, the integrand at $S = 0$ is just a constant and equal to $g_a(0)g_{b_1}(0)$. We obtain for small S

$$P(S) \approx S \cdot \frac{\pi}{2} g_a(0) g_{b_1}(0). \quad (12)$$

Thus in case of regular g's at zero value of x we always have linear level repulsion $P(S) \propto S$ for real, symmetric, random matrices, whose amplitude reflects the probability density $g_{a,b_1}(0)$ to find the matrix elements $a = 0$ and $b_1 = 0$ in the matrix $\mathbf{A}$. Higher-order corrections in S can be derived by Taylor expanding the g's around $x = 0$. We conclude that regular matrix element distribution functions g_{a,b_1} transform into regular level spacing distributions $P(S)$. From (11) and (12) we also notice that the level repulsion is **not** linear if $g_a(0)$ and $g_{b_1}(0)$ do not exist, i. e., if the distributions $g_a(x)$ and $g_{b_1}(x)$ are singular at $x = 0$, if there is infinite probability density for the matrix elements to have values $x = a = 0$ or $x = b_1 = 0$. We shall study this important case in Section 4.

3. General Non-Normal Real 2D Matrix Ensemble with Gaussian Distributed Matrix Elements

3.1. Level Spacing Distribution P(S): General

We start with (5) and observe that the dependence of the level distance S in the integrand of $P(S)$ on b_1 and b_2 is only through the product $B = b_1 b_2$. Therefore it is natural to introduce hyperbolic coordinates defined as

$$B = b_1 b_2, \quad \upsilon = \frac{b_1}{b_2}, \qquad \text{equivalent to } \; b_1^2 = B\upsilon, \quad b_2^2 = B/\upsilon. \quad (13)$$

Both B and υ run over the entire interval $(-\infty, \infty)$, but always have the same sign, $\operatorname{sgn} B = \operatorname{sgn} b_1 \cdot \operatorname{sgn} b_2 = \operatorname{sgn}\upsilon$. Positive B or υ indicate that both non-diagonal elements are positive or that both are negative. Negative values of the variables B and υ, instead, describe the case of non-diagonal elements with different sign. The Jacobian determinant is $J = 1/(2|\upsilon|)$, and for the area elements we have $\mathrm{d}b_1\,\mathrm{d}b_2 = \mathrm{d}B\,\mathrm{d}\upsilon/(2|\upsilon|)$.

To analyse the integral further we assume Gaussian distributed matrix elements, though with possibly different variances σ^2, σ_1^2, and σ_2^2, for a, b_1, and b_2, respectively.

$$g_a(a) = \frac{1}{\sigma\sqrt{\pi}} \exp\left(-\frac{a^2}{\sigma^2}\right), \qquad g_{b_i}(b_i) = \frac{1}{\sigma_i\sqrt{\pi}} \exp\left(-\frac{b_i^2}{\sigma_i^2}\right), \quad i = 1,2. \quad (14)$$

All three distributions are normalized to one. With this assumption the equation for $P(S)$ obtains the form

$$P(S) = \frac{1}{\sigma\sigma_1\sigma_2\sqrt{\pi}^3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\mathrm{d}a\,\mathrm{d}B\,\mathrm{d}\upsilon}{2|\upsilon|} \cdot \exp\left(-\left(\frac{a^2}{\sigma^2} + \frac{B\upsilon}{\sigma_1^2} + \frac{B}{\upsilon\sigma_2^2}\right)\right)\delta\left(S - 2\left|\sqrt{a^2 + B}\,\right|\right). \quad (15)$$

The integrand is even in the variable a, we thus can use $\int_{-\infty}^{\infty} \mathrm{d}a \to 2\int_0^{\infty} \mathrm{d}a$. Next, the level distance delta function does not depend on υ explicitly. But since the signs of υ and B are coupled, only the first and the third quadrant of the (B,υ)-plane contribute. In both these (B,υ)-quadrants it is $b_1^2 = B\upsilon = |B||\upsilon| \ge 0$ and $b_2^2 = B/\upsilon = |B|/|\upsilon| \ge 0$, guaranteeing the convergence of the (B,υ)-integrals. In the first quadrant we have positive B, while in the third one $-|B|$ is relevant for the level distance. This leads to

$$P(S) = \frac{2}{\sigma\sigma_1\sigma_2\sqrt{\pi}^3}\int_0^{\infty} \mathrm{d}a \exp\left(-\frac{a^2}{\sigma^2}\right) \cdot \int_0^{\infty} \mathrm{d}B \int_0^{\infty} \frac{\mathrm{d}\upsilon}{2\upsilon} \exp\left(-\frac{B\upsilon}{\sigma_1^2} - \frac{B}{\upsilon\sigma_2^2}\right) \cdot \left[\delta\left(S - 2\sqrt{a^2 + B}\right) + \delta\left(S - 2\left|\sqrt{a^2 - B}\right|\right)\right]. \quad (16)$$

Now all variables a, B, υ have to be integrated over positive values only. The sum of the delta functions then is independent of the variable υ and the υ-integral can be performed, cf. [17] No. 3.478,4:

$$\int_0^{\infty} \frac{\mathrm{d}\upsilon}{2\upsilon} \exp\left(-\frac{B}{\sigma_1^2}\upsilon - \frac{B}{\sigma_2^2}\frac{1}{\upsilon}\right) = K_0\left(\frac{2B}{\sigma_1\sigma_2}\right). \quad (17)$$

Here $K_0(x)$ is the modified Bessel function of second kind and zero-order. Its argument is always positive, $B \geq 0$. The level spacing distribution $P(S)$ becomes

$$P(S) = \frac{2}{\sigma\sigma_1\sigma_2\sqrt{\pi}^3}\int_0^\infty \mathrm{d}B\, K_0\left(\frac{2B}{\sigma_1\sigma_2}\right)\cdot G(B), \quad (18)$$

where $G(B)$ is the Gaussian averaged level distribution for fixed, given product $B \geq 0$ of the non-diagonal elements but varying diagonal elements a:

$$G(B) = \int_0^\infty \mathrm{d}a \exp\left(-\frac{a^2}{\sigma^2}\right) \cdot \left[\delta\left(S - 2\sqrt{a^2+B}\right) + \delta\left(S - 2\left|\sqrt{a^2-B}\right|\right)\right], \quad \text{with } B \geq 0. \quad (19)$$

The calculation of $G(B)$ is easy because of the delta functions and can be done analytically. Consider the first delta function. Its argument, denoted as $f(a) = S - 2\sqrt{a^2+B}$, with positive B, corresponds to real eigenvalues of the matrix $\mathbf{A}$. The zeros a_i of the delta function contribute to the integral only if they are real and positive. There is only one real, positive a_i, provided $B \leq S^2/4$. Then the variable $u = 4B/S^2$ fulfils $0 \leq u \leq 1$ and the f-zero reads $a_i = \frac{S}{2}\sqrt{1-u}$. The weight of the delta function contribution is given by the inverse of the derivative of f, which is $|f'(a_i)|^{-1} = \left(2\sqrt{1-u}\right)^{-1}$. The first delta function in $G(B)$ then leads to $G_{+re}(B) = \exp\left(-\frac{S^2}{4\sigma^2}(1-u)\right) / \left(2\sqrt{1-u}\right)$. The label $(+re)$ indicates that the term describes the case $+B$ and real (re) eigenvalues. Transforming the variable $B \to u$ this $G_{+re}(B)$ contributes the following term to the level spacing distribution:

$$P_{+re}(S) = \frac{S^2}{4\sigma\sigma_1\sigma_2\sqrt{\pi}^3}\int_0^1 \frac{\mathrm{d}u}{\sqrt{1-u}} \cdot K_0\left(\frac{S^2 u}{2\sigma_1\sigma_2}\right)\exp\left(-\frac{S^2}{4\sigma^2}(1-u)\right). \quad (20)$$

This integral has been considered already in [7] and leads to a level repulsion exponent $\rho = 2 - 0_{\log}$, see also Section 3.2. It will turn out that this contribution (20) is subdominant relative to the other two integrals for $P(S)$, which in contrast to (20) will lead to the repulsion exponent $\rho = 1$, cf. again Section 3.2.

We now calculate the $G(B)$ contributions coming from the second delta function. They are labelled by a $(-)$ sign (since $-B$ enters) and correspond to real as well as to imaginary eigenvalues of the matrix $\mathbf{A}$, depending on the size of B. They are labelled therefore by (re) if $a^2 - B \geq 0$ and by (im) if $a^2 - B < 0$. The relevant (positive) zeros a_i of the argument of the delta function are $a_i = \frac{S}{2}\sqrt{1+u}$ for all positive B or $u = 4B/S^2 \geq 0$ in case of (re) and $a_i = \frac{S}{2}\sqrt{u-1}$ for all positive B with $B \geq S^2/4$ and thus all $u = 4B/S^2 \geq 1$ in the case (im). The weights $|f'(a_i)|^{-1}$ of the delta function contributions are obtained from $|f'(a_i)| = 2\sqrt{1+u}$ for the case (re) and from $|f'(a_i)| = 2\sqrt{u-1}$ for (im). These formulae lead to the following two contributions to the level spacing distribution $P(S)$:

$$P_{-re}(S) = \frac{S^2}{4\sigma\sigma_1\sigma_2\sqrt{\pi}^3}\int_0^\infty \frac{\mathrm{d}u}{\sqrt{1+u}} \cdot K_0\left(\frac{S^2 u}{2\sigma_1\sigma_2}\right)\exp\left(-\frac{S^2}{4\sigma^2}(1+u)\right) \quad (21)$$

and

$$P_{-im}(S) = \frac{S^2}{4\sigma\sigma_1\sigma_2\sqrt{\pi}^3}\int_1^\infty \frac{\mathrm{d}u}{\sqrt{u-1}} \cdot K_0\left(\frac{S^2 u}{2\sigma_1\sigma_2}\right)\exp\left(-\frac{S^2}{4\sigma^2}(u-1)\right). \quad (22)$$

These three integrals (20), (21), and (22) can be summed up to give the complete level spacing distribution $P(S)$. To do this one introduces variables y such that in all three cases the exponential is $\exp(-y)$ with $y \geq 0$. In the case $(-im)$ one in addition substitutes $y \to -y$. The real eigenvalues, represented by equations (20) and (21), give y integrations from 0 to $S^2/4\sigma^2$ and from $S^2/4\sigma^2$ to ∞. The imaginary eigenvalues lead to an integral from 0 to ∞ or, equivalently, from $-\infty$ to 0. Respecting the always positive argument of K_0 one evaluates for the complete level spacing distribution function in closed form

$$P(S) = \frac{S}{2\sigma_1\sigma_2\sqrt{\pi}^3}\int_{-\infty}^\infty \frac{\mathrm{d}y}{\sqrt{|y|}} \cdot K_0\left(\frac{2\sigma^2}{\sigma_1\sigma_2}\left|y - \frac{S^2}{4\sigma^2}\right|\right)\exp(-|y|). \quad (23)$$

This formula has been reported to us independently by Professor H.-J. Sommers [18]. Because the integral converges also for $S = 0$, we can conclude from this formula that in leading order in S we have linear level repulsion due to the explicit factor S, namely

$$P(S) = S \cdot \frac{1}{\sigma_1\sigma_2\sqrt{\pi}^3}\int_0^\infty \frac{\mathrm{d}y}{\sqrt{y}} \cdot K_0\left(\frac{2\sigma^2}{\sigma_1\sigma_2}y\right)\exp(-y) + \text{h.o.t.} \quad (24)$$

The S-independent integral can be expressed in terms of the hypergeometric function F as will be shown in (33) of Section 3.2.

From (23) one might wish to work out also the higher-order terms in S, stemming from the integrand. However, if at small S one formally expands the factor K_0 in (23) in terms of an S^2 series, one obtains contributions to the integrand, which are no longer integrable. Therefore the small S behaviour of the y- or u-integrals is highly non-trivial. This agrees with our earlier observation in Section 2.2. We have to analyse that in detail by studying the individual integrals $P_{+re}(S)$, $P_{-re}(S)$, and $P_{-im}(S)$ of (20), (21), and (22), respectively.

3.2. Level Spacing Distribution P(S): Details of Small S Behaviour

As we have seen in Sections 2.2 and 3.1 for non-symmetric (and thus non-normal) matrices, the behaviour of $P(S)$ at small S is very delicate and depends on the details of the distribution functions $g_{a,b_i}(x)$ for the matrix elements. In order to achieve understanding we go back to (20) and make the substitutions $1-u \to u' \to u$ and $S^2u/(2\sigma_1\sigma_2) = u' \to u$ yielding

$$P_{+re}(S) = \frac{S}{2\sigma\sqrt{2\sigma_1\sigma_2}\sqrt{\pi}^3} \cdot \int_0^{\varepsilon} \frac{\mathrm{d}u}{\sqrt{u}} K_0(\varepsilon-u)\mathrm{e}^{-Au}. \tag{25}$$

Here we have introduced the notations

$$\varepsilon = \frac{S^2}{2\sigma_1\sigma_2}, \quad A = \frac{\sigma_1\sigma_2}{2\sigma^2}. \tag{26}$$

If ε is very small, $\varepsilon \ll 1$, we can use the approximation of $K_0(z)$ at small z, which is ([17], No. 8.447,1 and 3, and 8.362,3) $K_0(z) = -\ln\frac{z}{2} - C + O(z)$, where C is Euler's constant $0.577215\ldots$. Then the leading term in the above integral, also taking into account the Taylor expansion of e^{-Au}, can be written as

$$\hat{I}_{+re}(\varepsilon) = \int_0^{\varepsilon} \frac{\mathrm{d}u}{\sqrt{u}}(-\ln(\varepsilon-u)), \tag{27}$$

which after a simple substitution $\varepsilon - u = u' \to u$ can be found in [17] (No. 2.727,5). After the evaluation for small ε we get

$$\hat{I}_{+re}(\varepsilon) \approx 2\sqrt{\varepsilon}(-\ln\varepsilon) \propto S\ln S^{-2}, \tag{28}$$

meaning that including the explicit factor S in (25) we have

$$P_{+re}(S) \propto S^2 \ln S^{-2}. \tag{29}$$

The level repulsion exponent ρ for the case of positive B (which also guarantees real eigenvalues of the matrix $\mathbf{A}$) therefore is $\rho = 2 - 0_{\log}$. Positive B means that we consider Gaussian distributed non-normal, real 2D matrices, which have only positive or only negative non-diagonal matrix elements b_1, b_2. It is under this constraint that they have the rather strong repulsion exponent $\rho = 2 - 0_{\log}$, as reported already in [7].

The other two contributions (21) and (22) are different in behaviour. Indeed each of them gives rise to weaker level repulsion, i. e., to a smaller exponent ρ. They both exhibit linear level repulsion $P(S) \propto S$, thus $\rho = 1$. This dominates the stronger, quadratic repulsion for small S valid for $P_{+re}(S)$, so that the total $P(S)$ has linear level repulsion $P(S) \propto S$, as obtained in (23) and (24). The analytical reason for the quite different S dependence of $P_{+re}(S)$, from (20), in contrast to that of P_{-re} and P_{-im}, according to (21) and (22), is the **finite** range of the B or u integration in (20) versus the infinite integration intervals in the cases of negative B, namely (21) and (22). Because of these infinite integration intervals one cannot Taylor expand the K_0-function in the cases of P_{-re} and P_{-im}. Instead, its complete functional form including its tails affect the convergence and thus the S behaviour of the integrals in (21), (22).

In order to demonstrate this explicitly, we use the same substitution as before and arrive from (21) at the expression

$$P_{-re}(S) = \frac{S\exp\left(-\frac{S^2}{4\sigma^2}\right)}{2\sigma\sqrt{2\sigma_1\sigma_2}\sqrt{\pi}^3} \int_0^{\infty} \frac{\mathrm{d}u}{\sqrt{u+\varepsilon}} K_0(u)\mathrm{e}^{-Au}, \tag{30}$$

with the relevant integral

$$\hat{I}_{-re}(S) = \int_0^{\infty} \frac{\mathrm{d}u}{\sqrt{u+\varepsilon}} K_0(u)\mathrm{e}^{-Au}. \tag{31}$$

The meaning of ε and A is the same as in (26). The integrand decays exponentially for large u since also $K_0(u)$ does so. Thus the upper limit of the integral converges safely, independent of ε. For small u the integrand can be estimated in a similar way as before. The result is that the integral converges to a *finite* value for $\varepsilon \to 0$

and not to zero, as in (27) and (28) for $\hat{I}_{+re}(S)$, i. e., $\hat{I}_{-re}(S \to 0) \neq 0$ is finite as $S \to 0$. Consequently, from the explicit factor S in (30), we conclude that the level repulsion is linear, $\rho = 1$. Note that in contrast to the finite integration range in the integral (27), which for small S behaves $\propto S \log S^{-2}$, the infinite range integral (31) has a *finite*, non-zero limit for $S \to 0$.

It remains to estimate the contribution from the imaginary eigenvalues given by (22). Again, similar substitutions of the integration variable u lead us to the expression

$$P_{-im}(S) = \frac{S}{2\sigma\sqrt{2\sigma_1\sigma_2}\sqrt{\pi}^3}\int_0^\infty \frac{\mathrm{d}u}{\sqrt{u}} K_0(\varepsilon + u)\mathrm{e}^{-Au}. \tag{32}$$

ε and A are defined in (26). As before, the integral has a finite value, which does not go to zero with $\varepsilon \propto S^2 \to 0$, and thus the level repulsion of the contribution P_{-im} according to (22) and (32) is again linear, $\rho = 1$. In fact, the integral in (32) at $\varepsilon = 0$ can be calculated according to [17] (6.621,3) as

$$\int_0^\infty \frac{\mathrm{d}u}{\sqrt{u}} K_0(u)\mathrm{e}^{-Au} = \frac{\sqrt{\pi}^3}{\sqrt{1+A}} F\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{A-1}{A+1}\right), \tag{33}$$

where F is the hypergeometric function.

The conclusion is that the level repulsion of the complete level spacing distribution function $P(S)$ is linear, $\rho = 1$, due to the contributions $P_{-re}(S)$ and $P_{-im}(S)$, whereas the contribution $P_{+re}(S)$ alone has the level repulsion exponent $\rho = 2 - 0_{\log}$, as we have already found in [7] for the case of Gaussian random matrices with only positive or only negative non-diagonal elements. Note that the criterion for the different small S behaviour is, if the product $b_1 b_2 = B$ is always positive, $B \geq 0$, or if there are also $B < 0$ contributions. The eigenvalues according to this latter case are apparently more abundant and thus have less repulsion ($\propto S$), while the former ones with only a positive product $b_1 b_2$ lead to the repulsion behaviour $\propto S^2 \log S^{-2}$.

4. Symmetric (Normal) Real 2D Matrix Ensembles with Various Distributions of the Matrix Elements

In this section we treat 2D real random matrices which are *symmetric*, i. e., $b_1 = b_2 \equiv b$ and thus $B = b^2 \geq 0$ always (from here onwards we drop the labels 1 and 2 in $b_1 = b_2$). Such matrices are normal, i. e., the matrix commutates with its adjoint. The generalization here is, that we allow for a broad variety of matrix element distribution functions $g_{a,b}$. The following classes of matrix element distributions are considered: (1) Gaussian distribution revisited, (2) box (uniform) distribution, (3) Cauchy-Lorentz distribution, (4) exponential distribution, and (5) singular distribution (power law approaching zero value multiplied by exponential tail). In all these cases we shall start with equation (11) as a useful integral representation for $P(S)$ in terms of $g_{a,b}$.

4.1. Gaussian Distribution Revisited

This case of real, Gaussian distributed 2D matrices with possibly different widths of the diagonal and the non-diagonal matrix element statistics has been treated recently in [7]. We briefly repeat it here for the sake of completeness. Using the normalized Gaussians defined in (14) we immediately get

$$P(S) = \frac{S}{2\sigma_a\sigma_b}\mathrm{e}^{-\frac{S^2}{8}(\sigma_a^{-2}+\sigma_b^{-2})} I_0\left(\frac{S^2}{8}(\sigma_a^{-2} - \sigma_b^{-2})\right), \tag{34}$$

where $I_0(z)$ is the modified Bessel function of the first kind and zero-order. According to [17] (No. 8.447,1) its small-z expansion is $I_0(z) = 1 + \frac{z^2}{4} + \frac{z^4}{64} + O(z^6)$. Thus $I_0(0) = 1$ and the level repulsion is linear, $\rho = 1$. The details for the general case of different widths of the a and b statistics, $\sigma_a \neq \sigma_b$, has been analysed and discussed in [7]. We mention that in the case of equal statistics of all matrix elements $\sigma_a = \sigma_b = \sigma$ we get, of course, the well known 2D GOE result

$$P(S) = \frac{S}{2\sigma^2}\mathrm{e}^{-\frac{S^2}{4\sigma^2}}. \tag{35}$$

After normalizing the first moment to unity, $\langle S \rangle = 1$, leading to $\sigma = 1/\sqrt{\pi}$, the level spacing distribution $P(S)$ becomes the Wigner distribution $P_{\mathrm{Wigner}}(S) = \frac{\pi S}{2}\exp(-\pi S^2/4)$.

4.2. Box (Uniform) Distribution

Let us now consider very different matrix element distributions $g_{a,b}$. We start by studying the following uniform distributions for the matrix elements a and b,

$$g_a(a) = \frac{1}{2a_0}, \text{ if } |a| \leq a_0, \quad 0 \text{ otherwise}, \tag{36}$$

$$g_b(b) = \frac{1}{2b_0}, \text{ if } |b| \le b_0, \quad 0 \text{ otherwise} . \tag{37}$$

Thus the probability density product $g_a(a)g_b(b)$ is constant, equal to $(4a_0b_0)^{-1}$, inside the (centrally located) rectangle with sides $2a_0 \times 2b_0$, and therefore for S smaller than $2\min\{a_0, b_0\}$ the level spacing distribution $P(S)$ can be calculated exactly. It is equal to

$$P(S) = \frac{\pi S}{8a_0b_0}, \text{ if } S \le 2\min\{a_0, b_0\}. \tag{38}$$

Again the level repulsion is linear. This is consistent with our finding in Section 2.2 that whenever $g_{a,b}(0) = \text{const} \neq 0$ there is generically linear level repulsion $\rho = 1$. From this geometrical picture it is also clear that $P(S)$ is zero for $S \ge 2\sqrt{a_0^2 + b_0^2}$. For S in between $P(S)$ varies continuously.

Indeed, we can calculate $P(S)$ exactly for all S using (11). We observe that g_a and g_b enter this expression symmetrically. Therefore without loss of generality we assume that $b_0 \le a_0$, i. e., $\min\{a_0, b_0\} = b_0$. Then we have to consider two more intervals, namely (i) $2b_0 \le S \le 2a_0$ and (ii) $2a_0 \le S \le 2\sqrt{a_0^2 + b_0^2}$.

In the first case (i) the circle of radius $S/2$ intersects the rectangle at four points. The angle enclosed by the polar ray and the abscissa is equal to φ_0, where $S \sin\varphi_0 = 2b_0$. Therefore, the total length of the φ-interval contributing to the integral in (11) is just $4\varphi_0$, and consequently $P(S) = \frac{S}{4a_0b_0} \arcsin\frac{2b_0}{S}$.

In the second case (ii) the circle of radius $S/2$ intersects the rectangle at eight points, each pair of intersection points defining one φ-interval, where we get a contribution to the integral. But all four angles are the same due to the double reflection symmetry of the rectangle and the circle. The larger angle φ_2 between the polar ray and the abscissa is geometrically determined by $S \sin\varphi_2 = 2b_0$, and the smaller one by $S \cos\varphi_1 = 2a_0$. Thus for each pair the length of the φ-interval is equal to $\varphi_2 - \varphi_1$, and since there are four such intervals, the total length of the interval contributing to the integral is $4(\varphi_2 - \varphi_1)$.

Putting all together we obtain the following exact result for the level spacing distribution $P(S)$ in the case of the uniform (box) distributions $g_{a,b}$:

$$\begin{aligned} P(S) &= \frac{\pi S}{8a_0b_0}, \text{ if } S \le 2b_0 \le 2a_0, \\ &= \frac{S}{4a_0b_0} \arcsin\frac{2b_0}{S}, \text{ if } 2b_0 \le S \le 2a_0, \\ &= \frac{S}{4a_0b_0}\left(\arcsin\frac{2b_0}{S} - \arccos\frac{2a_0}{S}\right), \\ &\quad \text{if } 2a_0 \le S \le 2\sqrt{a_0^2+b_0^2}, \\ &= 0, \text{ if } S \ge 2\sqrt{a_0^2+b_0^2}. \end{aligned} \tag{39}$$

If instead of $b_0 \le a_0$ one has $b_0 \ge a_0$, simply interchange a_0 and b_0 in the above formulae.

4.3. Cauchy-Lorentz Distribution

The normalized probability densities for the matrix elements are defined by

$$g_a(a) = \frac{1}{\pi a_0\left(1 + \frac{a^2}{a_0^2}\right)}, \quad g_a(b) = \frac{1}{\pi b_0\left(1 + \frac{b^2}{b_0^2}\right)}. \tag{40}$$

From (11) and using plane polar coordinates we obtain

$$P(S) = \frac{S}{4\pi^2 a_0 b_0} \cdot \int_0^{2\pi} \frac{\mathrm{d}\varphi}{\left(1 + \frac{S^2}{4a_0^2}\cos^2\varphi\right)\left(1 + \frac{S^2}{4b_0^2}\sin^2\varphi\right)}. \tag{41}$$

The integral for $S \to 0$ gives 2π, so that at small S we have $P(S) \approx S/(2\pi a_0 b_0)$ in accordance with (12). But the integral (41) can also be done exactly. The result is

$$P(S) = \frac{S}{2\pi a_0 b_0} \cdot \frac{\alpha^2\sqrt{1+\beta^2} + \beta^2\sqrt{1+\alpha^2}}{(\alpha^2+\beta^2+\alpha^2\beta^2)\sqrt{1+\alpha^2}\sqrt{1+\beta^2}}, \tag{42}$$

where $\alpha^2 = S^2/(4a_0^2)$ *and* $\beta^2 = S^2/(4b_0^2)$. The asymptotic behaviour of $P(S)$ at large S, i. e., $\alpha^2 \gg 1$ and $\beta^2 \gg 1$, is an inverse quadratic power law:

$$P(S) \approx \frac{4(a_0+b_0)}{\pi S^2}. \tag{43}$$

If $a_0 = b_0 = a$, the complete formula for all level distances S reads

$$P(S) = \frac{S}{2\pi a^2} \cdot \frac{1}{\left(1 + \frac{\alpha^2}{2}\right)\sqrt{1+\alpha^2}}, \quad \text{with } \alpha^2 = S^2/(4a^2). \tag{44}$$

This expression for the level spacing statistics evidently mirror images the Cauchy-Lorentz distribution of the $(g_b = g_a)$ statistics in the $P(S)$ statistics.

It is interesting to note that $P(S)$ in (42) has a divergent (infinite) first moment, as is clearly seen from the asymptotics (43). A generalized power law statistics of the type $g_a(a) = C_a/(1+(a/a_0)^q)$, with $q = 4,6,\ldots$, however, has a finite first moment $\langle S\rangle < \infty$. It will be treated elsewhere.

4.4. Singular Times Exponential Distribution

Our normalized distributions in this subsection are chosen as

$$g_a(a) = C_a|a|^{-\mu_a}\mathrm{e}^{-\lambda_a|a|},\ g_b(b) = C_b|b|^{-\mu_b}\mathrm{e}^{-\lambda_b|b|}, \quad (45)$$

where the normalization constants are

$$C_i = \lambda_i^{1-\mu_i}/(2\Gamma(1-\mu_i)). \quad (46)$$

Here $i = a,b$, the exponents $\mu_i < 1$, and $\Gamma(x)$ is the gamma function. These distribution functions are singular but integrable power laws for $a,b \to 0$ and decay nearly exponentially in the tails. Using (11) and the reflection symmetry (evenness) of both distributions $g_{a,b}$ in (45) we get (note that S is positive only, $S \geq 0$)

$$P(S) = C_aC_bS\left(\frac{S}{2}\right)^{-(\mu_a+\mu_b)} \cdot \int_0^{\pi/2} \frac{\mathrm{d}\varphi \exp\left(-\frac{S}{2}(\lambda_a\cos\varphi + \lambda_b\sin\varphi)\right)}{\cos^{\mu_a}\varphi\sin^{\mu_b}\varphi}. \quad (47)$$

We did not succeed to calculate this integral analytically in closed form. However, one can evaluate it for small argument S, i. e., $S \to 0$, where the exponential can be approximated by 1 (equivalent to $\lambda_a,\lambda_b \to 0$, i. e., no tail effects in this small S range). The integral then is

$$\int_0^{\pi/2} \frac{\mathrm{d}\varphi}{\cos^{\mu_a}\varphi\sin^{\mu_b}\varphi} = \frac{\Gamma\left(\frac{1}{2}-\frac{\mu_a}{2}\right)\Gamma\left(\frac{1}{2}-\frac{\mu_b}{2}\right)}{2\Gamma\left(1-\frac{\mu_a}{2}-\frac{\mu_b}{2}\right)}. \quad (48)$$

From this we get the following level repulsion law, now being a fractional exponent power law:

$$P(S) = C_aC_bS\left(\frac{S}{2}\right)^{-(\mu_a+\mu_b)} \frac{\Gamma\left(\frac{1}{2}-\frac{\mu_a}{2}\right)\Gamma\left(\frac{1}{2}-\frac{\mu_b}{2}\right)}{2\Gamma\left(1-\frac{\mu_a}{2}-\frac{\mu_b}{2}\right)}. \quad (49)$$

The power law distribution for the matrix elements leads to a corresponding power law for the level spacing distribution. The power law exponents μ_a and μ_b of the matrix element distribution functions $g_{a,b}$ immediately transform into the level repulsion exponent. More precisely, the level repulsion exponent is $\rho = 1 - \mu_a - \mu_b$. We emphasize that $P(S)$ at $S = 0$ is integrable, if the matrix element distributions $g_a(a)$ and $g_b(b)$ are integrable at $a = 0$ and $b = 0$, respectively.

The physical interpretation of this repulsion exponent $\rho = 1 - \mu_a - \mu_b$ comprises two different possibilities, depending on the size of the singularity exponents μ_a and μ_b of the distribution functions g_a, g_b for the matrix elements. If the singularities are strong, more precisely, if $\mu_a + \mu_b > 1$, the repulsion exponent ρ is negative. This means that due to the rather strong sparsing of the matrix there is not a repulsion but, instead, an enhancement of the level distance $S = 0$, the zero eigenvalues are emphasized. If the singularities are weak, $\mu_a + \mu_b < 1$, the phenomenon of level repulsion remains, $\rho > 0$, despite the singularities for the matrix element distributions at $a = 0$ and $b = 0$.

The diagonal elements $\pm a$ and the non-diagonal elements b determine the level distance with equal weight since $S = 2\sqrt{a^2+b^2}$. It is for this reason that it is just the sum of the singularity exponents which determines ρ, giving equal weight to the singularities of the diagonal and non-diagonal elements to the repulsion. There are two typical limiting cases: (i) $\mu_a = \mu_b \equiv \mu$, both singularities are of equal strenght, or (ii) $\mu_a \equiv \mu_0 \neq 0$ and $\mu_b = 0$ or vice versa. Only one of the two matrix element distributions is singular while the other one remains regular. Then only the non-diagonal is sparsed and the diagonal is regular or the other way round. In the first case (i), if $\mu < 1/2$, we have still level repulsion, $\rho > 0$, while for $\mu > 1/2$ (but still less than 1), we find enhancement at zero distance $S = 0$. In the second case (ii) there will always be level repulsion despite the singularity at zero for either the non-diagonal or the diagonal distribution, since $\rho = 1 - \mu_0 > 0$ always. The repulsion exponent in this case will be between 0 and 1. Sparsing like this is one of the possible causes that the repulsion exponent scans the interval $(0,1)$.

These results are interesting in the context of quantum chaos of nearly integrable (KAM type) systems. As has been observed in a variety of different systems of mixed type, at small energies one finds the so-called fractional exponent power law level repulsion, well described by the Brody distribution rather than by the Berry-Robnik distribution [19 – 24], which in turn has been clearly demonstrated to apply at sufficiently large energies, i. e., at sufficiently small effective Planck constant. The observed deviation from the Berry-Robnik behaviour is due to the localization and tunneling effects, and is a subject of intense current research [25]. Phenomenologically it has been discovered in [13], further developed in [14, 15], and the connection with sparsed matrices was established in [16]. Quite generally, the matrix representation of a Hamilton operator of a nearly integrable (KAM type) system in the basis of the integrable part results in a sparsed banded matrix with non-zero diagonal elements [16]. Such a sparsed matrix is precisely characterized by the fact that many matrix elements are zero. In other words, the probability distribution function of the non-diagonal matrix elements $g_b(b)$ is singular at zero value of b, in a manner described by (45), while the diagonal matrix elements have a regular distribution function, which is precisely the case (ii) discussed above. Our 2D random matrix theory with such singular matrix element distribution functions therefore predicts qualitatively a fractional exponent power law level repulsion, the phenomenon observed in the above-mentioned works [13 – 16]. Thus we see that the study of random matrices with other than the invariant ensembles (GOE and GUE) is very important and connects to new physics. We leave this direction of research for further studies in the near future.

The large S behaviour of $P(S)$ in this case is obviously dominated by the exponentials. Although an exact solution of the integral (47) is not known, we shall show by applying the mean value theorem to the integral that $P(S)$ decays roughly exponentially at large $S \gg 1$. This will be analysed in the next subsection, devoted to pure exponential matrix element distributions without power law singularities at the origin.

4.5. Exponential Distribution

Here we start with the distribution functions of the previous subsection, but without singularities. I. e., we assume $\mu_a = \mu_b = 0$, yielding the case of purely exponential distribution of matrix elements:

$$g_a(a) = C_a \mathrm{e}^{-\lambda_a|a|}, \quad g_b(b) = C_b \mathrm{e}^{-\lambda_b|b|}, \quad \text{with } C_i = \lambda_i/2. \tag{50}$$

The level spacing distribution function in this case is

$$P(S) = C_a C_b S \int_0^{\pi/2} \mathrm{d}\varphi \exp\left(-\frac{S}{2}(\lambda_a \cos\varphi + \lambda_b \sin\varphi)\right). \tag{51}$$

We could not evaluate this analytically in closed form. For small S the linear level repulsion law with $\rho = 1$ is recovered, of course,

$$P(S) \approx \frac{\pi C_a C_b}{2} S = \frac{\pi \lambda_a \lambda_b}{8} S. \tag{52}$$

At large S we can estimate the integral (51) using the mean value theorem. First we write $\lambda_a \cos\varphi + \lambda_b \sin\varphi = \hat{A}\sin(\varphi + \phi)$, with $\phi = \arctan(\lambda_a/\lambda_b)$, and $\hat{A} = \sqrt{\lambda_a^2 + \lambda_b^2}$. We then substitute the integration variable from φ to $\chi = \varphi + \phi$, which now runs from ϕ to $\phi + \pi/2$. This transforms $P(S)$ into

$$P(S) = C_a C_b S \int_\phi^{\phi+\pi/2} \mathrm{d}\chi \, \mathrm{e}^{-\frac{S}{2}\hat{A}\sin\chi}. \tag{53}$$

Since the integrand is continuous and bounded we can apply the mean value theorem, saying that there is a value $\chi_0(S)$ in the interval between ϕ and $\phi + \pi/2$ such that the level spacing distribution is

$$P(S) = \frac{\pi C_a C_b}{2} \cdot S \cdot \mathrm{e}^{-\frac{S}{2}\hat{A}\sin\chi_0(S)}. \tag{54}$$

$\chi_0(S)$ is expected to be only weakly dependent on S. In this sense the tail of $P(S)$ is roughly exponential.

5. Triangular Matrix

A special, prototype case of a non-normal matrix $\mathbf{A}$ from (1) is the triangular matrix

$$\mathbf{A} = (A_{ij}) = \begin{pmatrix} a & b \\ 0 & -a \end{pmatrix}. \tag{55}$$

Such matrices have been studied in detail in [7]. For them $b_1 b_2 = B = 0$ and the level distance S does not depend on b. With the constraint $g_{b_2}(b_2) = \delta(b_2)$ in (5) the a and ($b_1 = b$) integrations factorize, leading to the level distribution function

$$\begin{aligned} P(S) &= \int_{-\infty}^{\infty} \mathrm{d}a \, \delta(S - 2|a|) g_a(a) \\ &= \frac{1}{2}\left[g_a\left(\frac{S}{2}\right) + g_a\left(-\frac{S}{2}\right)\right]. \end{aligned} \tag{56}$$

For even distributions $g_a(a)$ we have

$$P(S) = g_a\left(\frac{S}{2}\right), \; S \geq 0, \quad \text{normalized on } [0 \leq S \leq \infty). \tag{57}$$

Apparently for triangular matrices the level spacing distribution is the immediate mirror of the diagonal element distribution $g_a(a)$. If $g_a(a)$ is Gaussian, there is no level repulsion at all, the repulsion exponent is $\rho = 0$ (see already [7]), though still the S are Gaussian and not Poissonian distributed as in the integrable case. If $g_a(a)$ is Poissonian, so is $P(S)$. And if $g_a(a)$ is centered about some finite value $\bar{a}$, this also holds for the level spacing distribution. If the a distribution does not include the origin, there even is formally *infinite level repulsion*. For general matrices with non-zero mean values $\bar{a}$ and $\bar{b}_{1,2}$ the level spacing distribution is somewhat more tricky.

6. Discussion and Conclusions

We have studied the level spacing distribution of general 2D real random matrices. First general, non-normal random matrices with Gaussian distribution of the matrix elements have been considered, showing that in general the level repulsion is still linear, as for symmetric matrices. Under some constraints it is quadratic with logarithmic corrections. We have given an explicit formula for the level spacing distribution function in form of a threefold integral. Then we have considered symmetric matrices with general, other than Gaussian statistics for both the diagonal and the non-diagonal elements. We have shown that the level repulsion again is always linear provided the matrix element distribution functions are regular at zero value with finite, non-zero weight. We have explicitly considered the box-type (uniform), the Cauchy-Lorentz, the exponential times singular power law at zero, and the purely exponential matrix element distributions. Explicit closed form results for $P(S)$ have been obtained, except for the singular times exponential tail distributions, although we could present a very good understanding of the overall behaviour of $P(S)$ also in this case.

Our approach can obviously be extended to general 2D complex random matrices, first to Hermitian complex matrices, where similar general formulae like (11) can be obtained, by using spherical coordinates. Indeed, it becomes obvious that we shall always have quadratic level repulsion as long as the matrix element distributions $g(x)$ are regular at zero x, and a formula analogous to (12) will apply. Denoting the diagonal elements by $\pm a$, and the off-diagonal elements (complex conjugate) by $b \pm \mathrm{i}\, c$, we obtain

$$P(S) = \frac{S^2}{8} \int\int \cos\theta \,\mathrm{d}\theta \,\mathrm{d}\varphi \, g_a\left(\frac{S}{2}\sin\theta\right) \cdot g_b\left(\frac{S}{2}\cos\theta\sin\varphi\right) g_c\left(\frac{S}{2}\cos\theta\cos\varphi\right), \tag{58}$$

as claimed. The details of further analysis will be published in a separate paper.

We have also discussed the connection between the singular distributions of the matrix elements with a power law at zero value and the sparsed matrices which describe nearly integrable systems. In such systems one derives a fractional exponent power law level repulsion well known phenomenologically but poorly understood theoretically. Our approach and results promise new advances in this important direction of research in quantum chaos of mixed type systems initiated in [26] and general random matrix theory.

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